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On structure of CAT(0) groups

静岡大学理学部

保坂 哲也 (Tetsuya Hosaka)

We introduce some results on finitely generated groups of isometries of CAT(0) spaces and CAT(0) groups in [15].

Definition and detail of CAT(0) spaces are found in [3] and [9].

Let X be a metric space and let γ be an isometry of X. Then the *translation length* of γ is defined as $|\gamma| = \inf\{d(x,\gamma x) \mid x \in X\}$, and the *minimal set* of γ is defined as $\mathrm{Min}(\gamma) = \{x \in X \mid d(x,\gamma x) = |\gamma|\}$. An isometry γ is said to be *semi-simple* if $\mathrm{Min}(\gamma)$ is non-empty. Also an isometry γ is called

(1) *elliptic* if γ has a fixed point,

(2) *hyperbolic* if γ is semi-simple and not elliptic, and

(3) *parabolic* if γ is not semi-simple.

(cf. [3, Chapter II.6]).

In [15], we obtained the following theorem by observing the proof of [3, Theorem II.6.12].

Theorem 1. *Let X be a CAT(0) space and let Γ be a finitely generated group acting by isometries on X. If the center of Γ contains a hyperbolic isometry γ_0 of X, then there exist a normal subgroup $\Gamma' \subset \Gamma$, an element $\delta_0 \in \Gamma$ and a number $k_0 \in \mathbb{N}$ such that*

(i) $\Gamma = \Gamma' \rtimes \langle\delta_0\rangle$,

(ii) $\Gamma' \rtimes \langle\delta_0^{k_0}\rangle = \Gamma' \times \langle\gamma_0\rangle$ *is a finite-index subgroup of* Γ *and*

(iii) Γ/Γ' *is isomorphic to* $\mathbb{Z}$.

A *geometric* action on a CAT(0) space is an action by isometries which is proper ([3, p.131]) and cocompact. A group Γ is called a *CAT(0) group*, if Γ acts geometrically on some CAT(0) space. We note that every CAT(0) space on which some group acts geometrically is a proper space ([3, p.132]). Also we note that CAT(0) groups are finitely presented (cf. [3, Corollary I.8.11]).

For example, Bieberbach groups ([3, p.246], [4]), crystallographic groups ([4]), Coxeter groups and their torsion-free subgroups of finite-index ([6], [7], [19]) and fundamental groups of compact geodesic spaces of non-positive curvature ([3, p.159, p.237]) are CAT(0) groups. In particular, fundamental groups of Riemaniann manifolds of non-positive sectional curvature are CAT(0) groups. Moreover, M. W. Davis [6] has constructed a closed aspherical manifold of dimension $n \geq 5$ whose universal covering is not homeomorphic to $\mathbb{R}^n$ ([6], [8]). The fundamental groups of these exotic manifolds are also CAT(0) groups.

On structure of CAT(0) groups, we obtained the following theorem from Theorem 1 in [15].

Theorem 2. *Let Γ be a CAT(0) group. Then there exist subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n$, elements $\delta_{i+1}, \gamma_{i+1} \in \Gamma_i$ and $k_{i+1} \in \mathbb{N}$ for $i = 0, \dots, n-1$ such that*

(1) *γ_{i+1} is an element of the center of Γ_i with the order $o(\gamma_{i+1}) = \infty$ for $i = 0, \dots, n-1$,*
(2) *$\Gamma_i = \Gamma_{i+1} \rtimes \langle \delta_{i+1} \rangle$ for $i = 0, \dots, n-1$,*
(3) *$\Gamma_{i+1} \rtimes \langle \delta_{i+1}^{k_{i+1}} \rangle = \Gamma_{i+1} \times \langle \gamma_{i+1} \rangle$ is a finite-index subgroup of Γ_i,*
(4) *Γ_i / Γ_{i+1} is isomorphic to $\mathbb{Z}$ for $i = 0, \dots, n-1$,*
(5) *$\Gamma = (\cdots(((\Gamma_n \rtimes \langle \delta_n \rangle) \rtimes \langle \delta_{n-1} \rangle) \rtimes \langle \delta_{n-2} \rangle) \cdots) \rtimes \langle \delta_1 \rangle$,*
(6) *Γ_n has finite center, and*
(7) *$\Gamma_n \times A$ is a finite-index subgroup of Γ where $A = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_n \rangle$ which is isomorphic to $\mathbb{Z}^n$.*

Here, we introduce an easy example of a CAT(0) group.

Example. Let $\Gamma = \langle a, b \,|\, ab^2 = b^2 a \rangle$ and let $X = \mathbb{R}^2$ the euclidean plane. We consider the action of the group Γ on X defined by

$$a \cdot (x, y) = (x, y+1)$$
$$b \cdot (x, y) = (x+1, -y)$$

for any $(x, y) \in \mathbb{R}^2 = X$. Then $D = [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$ is a fundamental domain, $\Gamma D = X$ and Γ acts geometrically on X. Here we note that X/Γ is a

Klein bottle and the group Γ is a CAT(0) group which is the fundamental group of the Klein bottle.

Then $\gamma_0 := b^2$ is a center of the CAT(0) group Γ and a hyperbolic isometry of X. Here we obtain that

(i) $\Gamma = \langle a \rangle \rtimes \langle b \rangle$,

(ii) $\langle a \rangle \rtimes \langle b^2 \rangle = \langle a \rangle \times \langle b^2 \rangle$ is a finite-index subgroup of Γ which is isomorphic to $\mathbb{Z}^2$ and

(iii) $\Gamma / \langle a \rangle$ is isomorphic to $\mathbb{Z}$.

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